ALL-AROUND TEST / ORAL EXAM
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied and Computational Mathematics

1. Let

$$V_h = \{v : v|_{I_j} \in P^k(I_j) \qquad 1 \le j \le N\}$$

where

$$I_j = (x_{j-1}, x_j), \qquad\qquad 1 \le j \le N$$

with

$$x_j = jh, \qquad h = \frac{1}{N}.$$

Here $P^k(I_j)$ denotes the set of polynomials of degree at most k in the interval I_j.

Recall the L^2 projection of a function $u(x)$ into the space V_h is defined by the unique function $u_h \in V_h$ which satisfies

$$||u - u_h|| \le ||u - v|| \qquad \forall v \in V_h$$

where the norm is the usual L^2 norm. We assume $u(x)$ has at least $k+2$ continuous derivatives.

(1) Prove the error estimate

$$||u - u_h|| \le Ch^{k+1}$$

Explain how the constant C depends on the derivatives of $u(x)$.

(2) If another function $\varphi(x)$ also has at least $k+2$ continuous derivatives, prove

$$|\int_0^1 (u(x) - u_h(x))\varphi(x)dx| \le Ch^{2k+2}$$

Explain how the constant C depends on the derivatives of $u(x)$ and $\varphi(x)$.